

Purdue University

Purdue e-Pubs

Department of Computer Science Technical
Reports

Department of Computer Science

1992

On Domains of Superior Convergence of the SSOR Method over the SOR Method

A. Hadijidmos

Michael Neumann

Report Number:
92-031

Hadijidmos, A. and Neumann, Michael, "On Domains of Superior Convergence of the SSOR Method over the SOR Method" (1992). *Department of Computer Science Technical Reports*. Paper 953.
<https://docs.lib.purdue.edu/cstech/953>

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries.
Please contact epubs@purdue.edu for additional information.

**ON DOMAINS OF SUPERIOR CONVERGENCE OF THE
SSOR METHOD OVER THE SOR METHOD**

**A. Hadjidimos
M. Neumann**

**CSD-TR 92-031
May 1992**

ON DOMAINS OF SUPERIOR CONVERGENCE
OF THE SSOR METHOD OVER THE SOR
METHOD

A. Hadjidimos *

Department of Computer Sciences
Purdue University
West Lafayette, Indiana 47907

Michael Neumann †

Department of Mathematics
University of Connecticut
Storrs, Connecticut 06269-3009

*Research supported in part by NSF grant CCR-8619817 and by AFOSR grant 8810243.

†Research supported in part by NSF Grant DMS-8901860 and by NSF Grant DMS-9007030.

Abstract

Let ν denote the spectral radius of \mathcal{J}_B^A , the block Jacobi iteration matrix. For the classes of matrices (i) nonsingular M-matrices and (ii) p -cyclic consistently ordered matrices, we study domains in the (ν, ω) -plane, when $\nu < 1$, where the block SSOR iteration method has at least as favorable asymptotic rate of convergence as the block SOR method.

Let \mathcal{L}_ω^A and S_ω^A denote, respectively, the block SOR and SSOR iteration matrices. For the class of nonsingular M-matrices A we determine conditions when the spectral radii satisfy that

$$\rho(S_\omega^A) \leq \rho(\mathcal{L}_\omega^A), \quad \forall 0 < \omega \leq 2/(1 + \nu) \text{ and } \forall 0 \leq \nu < 1.$$

Under these conditions we also show that the optimal SOR iteration parameter is $\omega_b = 1$. For the class of p -cyclic consistently ordered matrices A we determine for which ω 's and ν 's,

$$\rho(S_\omega^A) < |\omega - 1| \quad (\leq \rho(\mathcal{L}_\omega^A)).$$

Our investigations use of the equality case in Wielandt's inequality between the spectral radii of a complex matrix and its nonnegative and irreducible majorizers and of Rouché's theorem for the location of zeros of complex functions.

1 INTRODUCTION

In this paper we seek to determine domains in the (ν, ω) -plane where the block *symmetric successive overrelaxation* (SSOR) *method* performs at least as favorably as the block *successive overrelaxation* (SOR) *method*. Here ν is the *spectral radius of the block Jacobi iteration matrix* and is assumed to lie in $[0, 1)$ and $\omega \in (0, 2)$ is the *relaxation parameter*.

Let A be an $n \times n$ real matrix and consider the partitioning of A into the $p \times p$ block matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \cdots & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ A_{p,1} & \cdots & \cdots & \cdots & A_{p,p} \end{pmatrix}. \quad (1.1)$$

Assume that all diagonal blocks in A are square and nonsingular. Let D be the block diagonal matrix given by $D = \text{diag}(A_{1,1}, \dots, A_{p,p})$. Then the *block Jacobi iteration matrix associated with A*, $\mathcal{J}_B^A = I - D^{-1}A$, admits the representation

$$\mathcal{J}_B^A = L + U, \quad (1.2)$$

where L and U are, respectively, a block strictly lower and a block strictly upper triangular matrices. For $\omega \in (0, 2)$, the *block SOR iteration matrix associated with A* is given by

$$\mathcal{L}_\omega^A = (I - \omega L)^{-1}[(1 - \omega)I + \omega U] \quad (1.3)$$

and the *block SSOR iteration matrix associated with A* is given by

$$\mathcal{S}_\omega^A = (I - \omega U)^{-1}[(1 - \omega)I + \omega L](I - \omega L)^{-1}[(1 - \omega)I + \omega U]. \quad (1.4)$$

When all the diagonal blocks of A are of order 1×1 , then the block Jacobi, SOR, and SSOR iteration matrices are termed the *point Jacobi*, the *point SOR* and the *point SSOR iteration matrices associated with A*, respectively.

Several results in the direction of finding where the block SSOR has a better asymptotic convergence rate than that of the SOR have already been considered in the literature for special classes of matrices. We mention here

two such results. (a) When A is a nonsingular M-matrix, Woźnicki [20] showed that

$$\rho(\mathcal{S}_\omega^A) \leq \rho(\mathcal{L}_\omega^A) < 1, \quad \forall \omega \in (0, 1) \text{ and } \forall \nu := \rho(\mathcal{J}_B^A) \in [0, 1). \quad (1.5)$$

Here $\rho(\cdot)$ denotes the spectral radius of a matrix. Woźnicki's proof was based on a generalization of a comparison theorem due to Varga [17]. (b) When A is a 3-cyclic irreducible H-matrix, Neumann [14] showed that for every $\tilde{\nu} := \rho(|\mathcal{J}_B^A|) \in (0, r_3)$, where $r_3 \approx 0.418192802$ is the unique positive root of the cubic

$$17r^3 + r^2 - r - 1 \quad (1.6)$$

in the interval $(0, 1)$, there is a neighborhood $\Omega_{\omega(A)}$ of

$$\omega(A) = \frac{2}{1 + \tilde{\nu}} \quad (1.7)$$

such that

$$\rho(\mathcal{S}_\omega^A) < |\omega - 1| \leq \rho(\mathcal{L}_\omega^A), \quad \forall \omega \in \Omega_{\omega(A)}. \quad (1.8)$$

The matrix A is called *block p-cyclic* if A has the additional property that

$$A = \begin{pmatrix} A_{1,1} & 0 & 0 & \cdots & A_{1,p} \\ A_{2,1} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{p-1,p-1} & \vdots \\ 0 & \cdots & \cdots & A_{p,p-1} & A_{p,p} \end{pmatrix}. \quad (1.9)$$

We mention that in this form, A is in *consistently ordered normal form*. When the transposed of A is considered, then A is in *inconsistently ordered normal form*, cf.[17, p.101]. Whether A is in consistently or inconsistently ordered normal form will have no bearing on some of our results here.

We intend to establish two principal results:

(i) Suppose that A is a nonsingular M-matrix so that \mathcal{J}_B^A is nonnegative. We shall show that if $\rho(\mathcal{L}_{\omega(A)}^A) = 1$, then

$$\rho(\mathcal{S}_\omega^A) \leq \rho(\mathcal{L}_\omega^A), \quad \forall \omega \in (0, \omega(A)], \quad (1.10)$$

and

$$\rho(\mathcal{L}_1^A) = \min_{\omega \in (0, 2)} \rho(\mathcal{L}_\omega^A). \quad (1.11)$$

This result is achieved by refining the analysis for studying the spectral radius of the SOR iteration matrix employed in Neumann [12] which uses the equality case in the inequality between the spectral radii of a complex matrix and a nonnegative and irreducible matrix which majorizes it due to Wielandt [19]. This is done in Section 2 under somewhat weaker assumptions than $\rho(\mathcal{L}_\omega^A)$ attaining the value 1 at $\omega = \omega(A)$.

(ii) From the work of Kahan [9] it is known that $\rho(\mathcal{L}_\omega^A) \geq |\omega - 1|$ and it is similarly shown in Young [21] that $\rho(\mathcal{S}_\omega^A) \geq (\omega - 1)^2$. Therefore it is reasonable to ask for which pairs (ν, ω) , with $\nu \in [0, 1]$ and $\omega \in (0, 2)$, does

$$\rho(\mathcal{S}_\omega^A) < |\omega - 1| ? \quad (1.12)$$

For any p -cyclic consistently ordered matrices, we fully characterize the entire set of (ν, ω) 's for which (1.12) holds. This is done in Section 3. The main tool that we use for this characterization is the application of Rouché's theorem, c.f. Tall [16] to the functional relationship between the eigenvalues λ of the SSOR and the eigenvalues μ of the Jacobi iteration matrices

$$[\lambda - (\omega - 1)^2]^p = \lambda[\lambda - (\omega - 1)]^{p-2}(2 - \omega)^2\omega^p\mu^p \quad (1.13)$$

which was found by Varga, Niethammer, and Cai [18]. The investigation here is in the spirit of earlier works of the authors [6, 7]

Particularly in Section 2 we shall use relatively standard material from the literature concerning nonnegative matrices, M-matrices, regular splittings, comparison theorems for regular splittings, etc. Therefore we shall not introduce these concepts and their properties in the paper, but rather refer the reader to the texts by Berman and Plemmons [2], Varga [17], and Young [21]. The only exception is Woźnicki's [20] extension of Varga's comparison theorem for regular splittings which we will cite in Section 2.

2 ON WHEN THE SSOR METHOD FOR M-MATRICES IS SUPERIOR IN THE ENTIRE DOMAIN $\{(\nu, \omega) | 0 \leq \nu < 1 \text{ and } 0 < \omega < 2/(1+\nu)\}$

Let A be an $n \times n$ nonsingular M-matrix, so that now the block Jacobi iteration matrix associated with A is nonnegative. As indicated in the introduction, one goal of this section is to show that if $\rho(\mathcal{L}_{\omega(A)}^A) = 1$, where $\omega(A) = 2/(1 + \rho(J_B^A)) = 2/(1 + \nu)$, then (1.10) and (1.11) hold. These will be established with a refinement of arguments used in [12]. Recall also our mentioning that for $\omega \in (0, 1]$, $\rho(\mathcal{S}_{\omega}^A) \leq \rho(\mathcal{L}_{\omega}^A)$, for all $\omega \in (0, 1]$, which follows from Woźnicki's comparison theorem [20, Thms. 12 and 13] which we shall cite now. (For a more accessible exposition of Woźnicki's results and further comparison theorems see Csordas and Varga [4].)

WOŹNICKI'S THEOREM *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2).$$

Furthermore, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$ hold, then a strict inequality holds in the inequality between the spectral radii.

Let $\omega \in (0, 2)$ and consider the block SOR operator given in (1.3). The block SOR majorizer is the matrix given by

$$\mathcal{H}_{\omega}^A = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]. \quad (2.1)$$

If ω is restricted to the interval $[1, 2)$, then on letting

$$\mathcal{U}_{\omega}^A = (\omega - 1)(I - \omega L)^{-1} \text{ and } \mathcal{V}_{\omega}^A = \omega(I - \omega L)^{-1}U, \quad (2.2)$$

we obtain the representations

$$\mathcal{H}_{\omega}^A = \mathcal{V}_{\omega}^A + \mathcal{U}_{\omega}^A \text{ and } \mathcal{L}_{\omega}^A = \mathcal{V}_{\omega}^A - \mathcal{U}_{\omega}^A. \quad (2.3)$$

Before we state the main result of this section, for the sake of completeness, recall Wielandt's theorem for comparing the spectral radius of a complex matrix with the spectral radius of a nonnegative and irreducible matrix which majorizes it. We quote from Varga [17, Lemma 2.3]. For the original paper see Wielandt [19]:

WIELANDT'S THEOREM Let R be a nonnegative and irreducible $n \times n$ matrix and let Q be an $n \times n$ complex matrix with $|Q| \leq R$. If β is any eigenvalue of Q , then

$$|\beta| \leq \rho(R). \quad (2.4)$$

Moreover, equality is valid in (2.4), i.e., $\beta = \rho(R)e^{i\phi}$, if and only if $|Q| = R$, and where Q has the form

$$Q = e^{i\phi} E R E^{-1}, \quad (2.5)$$

and E is a diagonal matrix whose diagonal entries have modulus unity.

We are now ready to state:

THEOREM 2.1 Let A be an $n \times n$ nonsingular M -matrix. If there is a value $\omega_0 \in (1, 2)$ for which $\rho(\mathcal{L}_{\omega_0}^A) = \rho(\mathcal{H}_{\omega_0}^A)$, then

$$\rho(\mathcal{L}_\omega^A) = \rho(\mathcal{H}_\omega^A), \quad \forall \omega \in (0, 2), \quad (2.6)$$

and

$$\rho(\mathcal{S}_\omega^A) \leq \rho(\mathcal{L}_\omega^A), \quad \forall \omega \in (0, \omega(A)], \quad (2.7)$$

where $\omega(A)$ is given in (1.7). Moreover,

$$\rho(\mathcal{L}_1^A) = \min_{\omega \in (0, 2)} \rho(\mathcal{L}_\omega^A). \quad (2.8)$$

Proof: Suppose that $\omega_0 \in (1, 2)$ is an arbitrary but fixed value for which $\rho(\mathcal{L}_{\omega_0}^A) = \rho(\mathcal{H}_{\omega_0}^A)$. Let P be a permutation matrix which transforms $\mathcal{H}_{\omega_0}^A$ via the similarity $P\mathcal{H}_{\omega_0}^A P^T$ to a Frobenius block triangular normal form. Note that because in a permutation similarity diagonal entries migrate to diagonal entries, the expansion of the expression for $\mathcal{U}_{\omega_0}^A$ in a Neumann series shows that all diagonal entries of $\mathcal{S}_{\omega_0}^A$, and hence all the diagonal entries in its Frobenius normal form, are positive. Moreover, all diagonal blocks in the Frobenius normal form of $\mathcal{H}_{\omega_0}^A$ are irreducible matrices. As $|P\mathcal{L}_{\omega_0}^A P^T| \leq P\mathcal{H}_{\omega_0}^A P^T$ because $|\mathcal{L}_{\omega_0}^A| \leq \mathcal{H}_{\omega_0}^A$ and as $\rho(\mathcal{L}_{\omega_0}^A) = \rho(\mathcal{H}_{\omega_0}^A)$, $P\mathcal{L}_{\omega_0}^A P^T$ must have at least one diagonal block whose spectral radius equals the spectral radius of the corresponding irreducible diagonal block in $P\mathcal{H}_{\omega_0}^A P^T$. Let $\mu = \mu(\omega_0)$ represent an index of a diagonal block of $P\mathcal{L}_{\omega_0}^A P^T$ for which this equality occurs. It follows by the case of equality in the inequality in Wielandt's Theorem, that for each eigenvalue of $\lambda = \lambda_\mu$ of $(P\mathcal{L}_{\omega_0}^A P^T)_{\mu, \mu}$ for which $|\lambda| = \rho((P\mathcal{H}_{\omega_0}^A P^T)_{\mu, \mu})$, there exists a $\phi = \phi_\lambda$ and a diagonal matrix $E = E_\lambda$,

with $|E| = I$, such that $e^{i\phi} E(P\mathcal{L}_{\omega_0}^A P^T)_{\mu,\mu} E^{-1} = (P\mathcal{H}_{\omega_0}^A P^T)_{\mu,\mu}$. But then from (2.3) we see that

$$e^{i\phi} E[P(\mathcal{V}_{\omega_0}^A - \mathcal{U}_{\omega_0}^A)P^T]_{\mu,\mu} E^{-1} = [P(\mathcal{V}_{\omega_0}^A + \mathcal{U}_{\omega_0}^A)P^T]_{\mu,\mu} \quad (2.9)$$

from which, because $|x - y| \geq x + y$ implies $|x - y| = x + y$ for nonnegative numbers x and y , it follows that

$$|(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu} - (P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}| = (P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu} + (P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}. \quad (2.10)$$

The last equation gives the elementwise equalities

$$\begin{aligned} & |[(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} - [(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l}| \\ &= [(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} + [(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} \end{aligned} \quad (2.11)$$

for all entries $(k, l) \in \bar{\mu}$, where $\bar{\mu}$ is the subset of the indices in $\{1, \dots, n\}$ determined by the μ -th diagonal block in the Frobenius normal form of $\mathcal{H}_{\omega_0}^A$. But as the matrices $P\mathcal{V}_{\omega_0}^A P^T$ and $P\mathcal{U}_{\omega_0}^A P^T$ are both nonnegative, (2.11) holds if and only if

$$\begin{cases} [(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} > 0 \Rightarrow [(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} = 0, \\ [(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} > 0 \Rightarrow [(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} = 0. \end{cases} \quad (2.12)$$

We have already mentioned that the Neumann expansion applied to \mathcal{U}_{ω}^A together with the properties of permutation similarities easily yield that all diagonal entries of $(P\mathcal{U}_{\omega}^A P^T)_{\mu,\mu}$ are positive and equal to $\omega - 1$. We therefore deduce from (2.9) and (2.12) that $e^{i\phi} = -1$. (This was the conclusion made in [12] under more stringent conditions on A and for $\omega_0 = \omega(A)$ to which we referred in the introduction). Equality (2.9) gives now that

$$E(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu} E^{-1} - E(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu} E^{-1} = (P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu} + (P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}. \quad (2.13)$$

Taking into account that E is a diagonal matrix, (2.12) has now the implication that:

$$\begin{cases} [(P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} > 0 \Rightarrow E_{k,k}(E^{-1})_{l,l} = -1 \\ [(P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu}]_{k,l} > 0 \Rightarrow E_{k,k}(E^{-1})_{l,l} = 1. \end{cases} \quad (2.14)$$

Another implication which (2.12) has upon expanding the expressions for \mathcal{U}_ω^A and \mathcal{V}_ω^A of (2.2) in Neumann series as applicable is that

$$(P\mathcal{U}_\omega^A P^T)_{\mu,\mu} \stackrel{\varepsilon}{=} (P\mathcal{U}_{\omega_0}^A P^T)_{\mu,\mu} \text{ and } (P\mathcal{V}_\omega^A P^T)_{\mu,\mu} \stackrel{\varepsilon}{=} (P\mathcal{V}_{\omega_0}^A P^T)_{\mu,\mu}, \quad \forall \omega \in (1, 2), \quad (2.15)$$

where the symbol $\stackrel{\varepsilon}{=}$ designates matrices that are combinatorially identical, that is, their nonzero entries occur in same locations. But then (2.13)–(2.15) yield that

$$E(P(\mathcal{U}_\omega^A - \mathcal{V}_\omega^A)P^T)_{\mu,\mu} E^{-1} = (P(\mathcal{U}_\omega^A + \mathcal{V}_\omega^A)P^T)_{\mu,\mu}, \quad \forall \omega \in (1, 2). \quad (2.16)$$

This together with (2.3) gives that $\rho(\mathcal{L}_\omega^A) = \rho(\mathcal{H}_\omega^A)$ for all $\omega \in (1, 2)$. As $\mathcal{L}_\omega^A = \mathcal{H}_\omega^A$ for all $\omega \in (0, 1]$, the proof of (2.6) is now complete.

We next prove (2.7). We know from Woźnicki's Theorem, that $\rho(S_\omega^A) \leq \rho(\mathcal{L}_\omega^A)$ for all $\omega \in (0, 1]$. Thus we only need consider ω 's in $(1, \omega(A)]$. To this end consider the SSOR majorizer given by

$$\mathcal{T}_\omega^A = (I - \omega U)^{-1}[(\omega - 1)I + \omega L](I - \omega L)^{-1}[(\omega - 1)I + \omega U] \quad (2.17)$$

and SOR majorizers given in and (2.1). From the works of Alefeld and Varga [1] and Külsch [10] we know that for $\omega \in (1, 2)$, both majorizers are iteration matrices induced by regular splittings of the matrices

$$A_\omega = \frac{2 - \omega}{\omega} I - \mathcal{J}_B^A. \quad (2.18)$$

which have nonpositive off-diagonal entries. It is further known that for each $\omega \in (1, \omega(A))$, A_ω is a nonsingular M-matrix; for $\omega = \omega(A)$, A_ω is a singular M-matrix; while for $\omega > \omega(A)$, A_ω is not an M-matrix. The SSOR majorizer operator is induced by the splitting

$$\begin{aligned} A_\omega &= \frac{1}{\omega^2}(I - \omega L)(I - \omega U) - \frac{1}{\omega^2}[(\omega - 1)I + \omega L][(\omega - 1)I + \omega U] \\ &=: (M_1)_\omega - (N_1)_\omega, \end{aligned} \quad (2.19)$$

while the SOR majorizer operator is induced by the splitting

$$A_\omega = \frac{I - \omega L}{\omega} - \frac{[(\omega - 1)I + \omega U]}{\omega} =: (M_2)_\omega - (N_2)_\omega. \quad (2.20)$$

As $\omega > 1$ and $(I - \omega U)^{-1} \geq I$, it is readily checked that for all $\omega \in (1, \omega(A))$,

$$[(M_1)_\omega]^{-1} \geq [(M_2)_\omega]^{-1}. \quad (2.21)$$

Thus on applying Woźnicki's Theorem to the iteration matrices induced by the splittings (2.19) and (2.20) and taking into account our result in (2.6) we obtain that

$$\begin{aligned}\rho(\mathcal{L}_\omega^A) &= \rho(\mathcal{H}_\omega^A) = \rho([(M_2)_\omega]^{-1}(N_2)_\omega) \\ &\geq \rho([(M_1)_\omega]^{-1}(N_1)_\omega) = \rho(\mathcal{T}_\omega^A) \geq \rho(\mathcal{S}_\omega^A), \quad \forall \omega \in (1, \omega(A)).\end{aligned}$$

Applying further continuity arguments to the spectral radius at $\omega(A)$ gives now that (2.7) is true.

We come now to the proof of (2.8). That $\rho(\mathcal{L}_\omega^A) \leq \rho(\mathcal{L}_1^A)$ for all $\omega \in (0, 1]$ is proved in [17, Theorem 3.16]. Next, as for each $\omega > \omega(A)$, the matrix A_ω , whose off-diagonal entries are nonpositive, is not a nonsingular M-matrix, it follows from Varga's celebrated regular splittings theorem [17, Theorem 3.13] and from the properties of nonsingular M-matrices that the spectral radius of the iteration matrix \mathcal{H}_ω^A which induced by the regular splitting (2.19) must satisfy that $\rho(\mathcal{H}_\omega^A) \geq 1$. Thus, by (2.6), we have that

$$\rho(\mathcal{L}_\omega^A) \geq 1.$$

Suppose now that $\omega \in [1, \omega(A))$ and note that (2.6) allows us to work with the spectral radii of \mathcal{H}_ω^A instead of the spectral radii of \mathcal{L}_ω^A . As the function

$$\frac{\omega}{2 - \omega}$$

is increasing in $[1, \omega(A))$, we can derive from (2.18) that for the matrices

$$(A_\omega)^{-1} = \frac{\omega}{2 - \omega} [I + \frac{\omega}{2 - \omega} \mathcal{J}_B^A + \dots],$$

$$0 \leq (A_{\omega_1})^{-1} \leq (A_{\omega_2})^{-1}, \quad \forall 1 \leq \omega_1 \leq \omega_2 < \omega(A). \quad (2.22)$$

Moreover, from (2.20) we see that

$$(N_2)_{\omega_1} \leq (N_2)_{\omega_2}, \quad \forall 1 \leq \omega_1 \leq \omega_2 < \omega(A). \quad (2.23)$$

Thus from the well known formula of Varga [17, Eq. (3.75)] representing the spectral radius of an iteration matrix induced from a regular splitting we obtain using both (2.22) and (2.23) that

$$\begin{aligned}\rho(\mathcal{H}_{\omega_1}) &= \rho((A_{\omega_1})^{-1}N_{\omega_1}) / (1 + \rho((A_{\omega_1})^{-1}N_{\omega_1})) \\ &\leq \rho((A_{\omega_2})^{-1}N_{\omega_2}) / (1 + \rho((A_{\omega_2})^{-1}N_{\omega_2})) = \rho(\mathcal{H}_{\omega_2})\end{aligned}$$

for all $1 \leq \omega_1 \leq \omega_2 < \omega(A)$. Hence, using (2.6) together with continuity arguments at $\omega(A)$, we see that on the interval $[1, \omega(A)]$,

$$\rho(\mathcal{L}_1^A) = \min_{\omega \in [1, \omega(A)]} \rho(\mathcal{L}_\omega^A).$$

This concludes the proof (2.8) and of the entire theorem. \square

An immediate consequence of Theorem 2.1 is the following:

COROLLARY 2.1 *Suppose A is a nonsingular M-matrix for which $\rho(\mathcal{L}_{\omega(A)}^A) = 1$. Then (2.6)–(2.8) hold.*

Proof: As is commonly known, but has been explicitly used in [12], $\rho(\mathcal{H}_{\omega(A)}^A) = 1$. The conclusion follows now taking ω_0 of Theorem 2.1 to equal $\omega(A)$. \square

Remark 2.1: Corollary 2.1 points to what lead us to believe that (2.7) is true. On the one hand cases for which for a nonsingular M-matrix, $\rho(\mathcal{L}_{\omega(A)}^A)$ attains 1 have been investigated in the literature, though we only know to specifically cite here [12]. On the other hand, it was shown in [13] that for any nonsingular M-matrix A , $\rho(\mathcal{S}_{\omega(A)}^A) < 1$. In connection with bounds on the relaxation parameter for which the SSOR method is convergent for M- and H-matrices we refer the reader to Neumaier and Varga [11] and Hadjidimos and Neumann [5]. Finally, note that the proof of Theorem 2.1 shows that under the conditions of the theorem, for $\omega \in (1, 2)$, $\rho(\mathcal{L}_\omega^A)$ attains $\rho(\mathcal{H}_\omega^A)$ only on a negative eigenvalue on its spectral circle, a fact which was also concluded in [12] under more restrictive assumptions. The following

graph:

FIGURE 1

illustrates some of the results of Theorem 2.1 for the 7×7 primitive M-matrix:

$$A = \begin{pmatrix} 1 & -.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -.8 & 0 & 0 \\ -.8 & 0 & 0 & 0 & 1 & -.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -.8 \\ -.8 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here $\nu \approx 0.898986$. On taking E to be the 7×7 diagonal matrix whose i -th diagonal entry is given by $(-1)^i$, it is readily checked that for any $\omega \in (1, 2)$, $-E\mathcal{L}_\omega^A E^{-1} \approx \mathcal{H}_\omega^A$, so that $\rho(\mathcal{L}_\omega^A) = \rho(\mathcal{H}_\omega^A)$ for any such ω and all other conclusions of the theorem apply. For example and as illustrated in Figure 1, the spectral radius of the SOR matrix is smallest at $\omega = 1$ and it attains 1 at $\omega(A) = 2/(1 + \nu) \approx 1.053194$. We mention that under the conditions of the theorem, the optimal ω for the block SSOR iteration matrix can well be greater than 1 as can be seen in the figure. We further comment that although (2.21) holds for $(M_1)_\omega$ and $(M_2)_\omega$ of (2.19) and (2.20), respectively, it is not in general true that for these matrices, $(M_1)_\omega^{-1} > (M_2)_\omega^{-1}$. Therefore even if A is a nonsingular and irreducible M-matrix so that $A^{-1} > 0$, we can not use the latter part of Woźnicki's Theorem to strengthen the conclusion of (2.7) under this stronger assumption.

Remark 2.2 There are known families of M-matrices which satisfy the conditions of Theorem 2.1. For example, in [12] it is shown that if A is a p -cyclic, $p = 2k - 1 \geq 3$, irreducible inconsistently ordered M-matrix, then $\rho(\mathcal{L}_{\omega(A)}^A) = 1$. We further mention that Nichols and Fox [15] show that for a p -cyclic matrix A which is not consistently ordered and not necessarily an M-matrix or an irreducible matrix, but whose $(\mathcal{J}_B^A)^p$ has a nonnegative spectrum and $\nu < 1$, the optimal SOR relaxation parameter is equal to 1. Thus certain subclasses of M-matrices which satisfy the conditions of our Theorem 2.1 also satisfy the conditions of [15]. For these subclasses of M-matrices the results of Nichols and Fox and Theorem 2.1 are in agreement, but via different proofs. We finally comment that the satisfaction of the "nonoverlapping" condition (2.12) for the entries of \mathcal{V}_ω^A and \mathcal{U}_ω^A in same locations for all $\omega \in (0, 2)$ is insufficient for the condition (2.6) in Theorem 2.1 to hold. This is born out by a careful examination of the conclusions of [12, Theorem 3.5(ii)] for the case when A is an irreducible nonsingular

inconsistenly ordered p -cyclic, $p = 2k \geq 4$, M-matrix.

3 REGIONS OF DOMINANT CONVERGENCE OF THE SSOR METHOD FOR GENERAL p - CYCLIC MATRICES

Let A be a p -cyclic matrix with nonsingular diagonal blocks as in (1.9). In this section we shall inquire after regions in the (ν, ω) -plane where the superiority of the asymptotic rate of convergence of the block SSOR method is guaranteed over the asymptotic rate of convergence of the block SOR method because the leftmost inequality in the separation between the spectral radii

$$\rho(S_\omega^A) < |\omega - 1| \leq \rho(\mathcal{L}_\omega^A) \quad (3.1)$$

holds ?

As in the proof of our main result of this section we shall require Rouché's theorem, let us quote its statement as it appears in Tall's book [16]:

ROUCHÉ'S THEOREM: *Suppose G and F are analytic functions in a domain containing the track and the interior of a close Jordan contour Γ described anti-clockwise. If*

$$|F(\lambda) - G(\lambda)| < |G(\lambda)|, \quad \forall \lambda \in \Gamma,$$

then $G(\lambda)$ and $F(\lambda)$ have the same number of zeros inside Γ .

Let $\lambda \in \sigma(S_\omega^A)$ and $\mu \in \sigma(\mathcal{J}_B^A)$, where $\sigma(\cdot)$ denotes the spectrum of a matrix. We begin by defining a functional equation which is motivated by the functional relation (1.13)

$$g(\lambda) := [\lambda - (\omega - 1)^2]^p - \lambda[\lambda - (\omega - 1)]^{p-2}(2 - \omega)^2\omega^p\mu^p = 0. \quad (3.2)$$

We comment that it can be shown, c.f., Chong and Cai [3] and [7], that for A given in (1.9),

$$\sigma(S_\omega^A) = \sigma(S_\omega^{A^T}), \quad \forall \omega \in (0, 2).$$

Therefore, although we state all our results in this section for A in the p -cyclic consistently ordered normal form, these results hold also for A in the p -cyclic inconsistently ordered normal form.

Suppose that $\omega \neq 1$. For $\lambda \in C$ set $\xi = \lambda/(\omega - 1)$ and observe that $|\lambda| < |\omega - 1|$ if and only if $|\xi| < 1$. A substitution of λ in terms of ξ and ω

in (3.2) and some algebraic simplifications with the additional substitution $t := (\omega - 1)$ yield the equation

$$F(\xi) := \frac{g(t\xi)}{t^{p-1}} = t(\xi - t)^p - \xi(\xi - 1)^{p-2}(1 - t)^2(1 + t)^p \mu^p. \quad (3.3)$$

Note that $t \in (-1, 1) \setminus \{0\}$. Next define the function

$$G(\xi) := t(\xi - t)^p. \quad (3.4)$$

Since for any permissible t , $G(\xi)$ has all its roots in the interior of the unit circle we ask, in view of Rouché's theorem: **Given a permissible t and denoting the unit disc by \mathcal{D} and its boundary by $\partial\mathcal{D}$, for which $\mu \in C$ does it hold that**

$$|\xi(\xi - 1)^{p-2}(1 - t)^2((1 + t)^p \mu^p| < |t(\xi - t)^p|, \quad \forall \xi \in \partial\mathcal{D} ? \quad (3.5)$$

This question is equivalent to the question of determining for a given permissible t , for which μ 's in C does it hold that

$$\min_{\xi \in \partial\mathcal{D}} \frac{|\xi - t|^p}{|\xi - 1|^{p-2}} > \frac{(1 - t)^2(1 + t)^p}{|t|} |\mu|^p ? \quad (3.6)$$

Since $\xi \in \partial\mathcal{D}$, $\xi = x + iy$, where $x, y \in R$ and $x^2 + y^2 = 1$. After some further elementary algebraic manipulations, the question posed in (3.6) can be recast as for which values of $\mu \in C$,

$$\min_{-1 \leq x \leq 1} \frac{(1 + t^2 - 2tx)^{p/2}}{(1 - x)^{p/2-1}} > \frac{2^{p/2-1}(1 - t)^2(1 + t)^p}{|t|} |\mu|^p ? \quad (3.7)$$

We are now ready to prove the main result of this section.

THEOREM 3.1 *Let A be a nonsingular block p -cyclic, $p \geq 3$, matrix given in (1.9). Let \mathcal{J}_B^A , \mathcal{L}_ω^A , and S_ω^A be, respectively, the block Jacobi, the block SOR, and the block SSOR iteration matrices. Suppose that $\nu = \rho(\mathcal{J}_A^B)$. Set*

$$\omega_p^* = \frac{2\sqrt{p}}{\sqrt{2} + \sqrt{p}}. \quad (3.8)$$

Then for each point (ω, ν) in the domain

$$\mathcal{R}(p) = \begin{cases} 0 < \omega \leq \omega_p^*, & 0 \leq \nu < |1 - \omega|^{1/p} / 2^{1-2/p} (2 - \omega)^{2/p} \\ \omega_p^* \leq \omega < 2, & 0 \leq \nu < p^{1/2} (\omega - 1)^{1/2} / 2^{1/p} (p - 2)^{1/2-1/p} \omega \end{cases}, \quad (3.9)$$

in the (ω, ν) -plane, the inequality (3.1) is valid.

Proof: In accordance with the left-hand-side of (3.7) define the function

$$h(x, t) = \frac{(1 + t^2 - 2tx)^{p/2}}{(1 - x)^{p/2-1}}. \quad (3.10)$$

We shall use the symbol " \sim " to denote equality in sign between two expressions. Then partial differentiating $h(x, t)$ with respect to x and omitting all possible positive expressions which appear as multipliers which are encountered during simplifying the expression we obtain that

$$\begin{aligned} \partial h / \partial x \sim & -pt(1 - x)^{p/2-1}(1 + t^2 - 2tx)^{p/2-1} + \\ & (p/2 - 1)(1 + t^2 - 2tx)^{p/2}(1 - x)^{p/2-2} \sim \dots \sim \\ & t\{x - (1/4)[-(p - 2)(t + 1/t) + 2p]\}. \end{aligned} \quad (3.11)$$

Recall that $t = \omega - 1$ and $t \neq 0$. We distinguish between two cases: $t < 0$ and $t > 0$.

Case 1: $-1 < t < 0$. In this case, for all $p \geq 3$, it can be easily verified that $t + 1/t < -2$ which, together with the fact that $-1 \leq x \leq 1$, gives that $x - (1/4)[-(p - 2)(t + 1/t) + 2p] \leq -1$. This inequality makes the rightmost expression in (3.11) positive. Hence the minimum of $h(x, t)$ occurs at $x = -1$. For $p = 2$, it can readily be verified that $x = -1$ also minimizes $h(x, t)$. Substituting $x = -1$ in (3.7) shows that the set of all μ 's in C which satisfy it is given by

$$|\mu| < \frac{|t|^{1/p}}{2^{1-2/p}(1 - t)^{2/p}} = \frac{|1 - \omega|^{1/p}}{2^{1-2/p}(2 - \omega)^{2/p}}. \quad (3.12)$$

Case 2: $0 < t < 1$. In this case, for all $p \geq 2$, it can be readily checked that $t + 1/t > 2$ which, together with the fact that $-1 \leq x \leq 1$, gives that $s(t, p) := (1/4)[-(p - 2)(t + 1/t) + 2p] \leq 1$. For $p = 2$ we have equality in this inequality, otherwise $s(t, p) < 1$. For $p \geq 3$, the stronger stipulation $s(t, p) \leq -1$ holds if and only if

$$1 < \omega \leq \omega_p^*, \quad (3.13)$$

while $-1 \leq s(t, p) < 1$ if and only if

$$\omega_p^* \leq \omega < 2. \quad (3.14)$$

The inequality $s(t, p) \leq -1$ makes the rightmost expression in (3.11) nonnegative for all $x \in [-1, 1]$. This means that the minimum of $h(x, t)$

occurs again at $x = -1$. Substituting $x = -1$ in (3.7) shows that the set of all μ 's in C which satisfy (3.7) is the same as the set of μ 's which satisfy (3.12).

We next consider the situation when ω is in the range given in (3.14). In this case for every $t = \omega - 1$ and every $p \geq 3$ substituting the value $x_{t,p} = s(t,p)$ in the last sign equivalent expression to $\partial h / \partial x$ given in (3.11) causes it to vanish. Prior to the point $x_{t,p}$, one can check that $\partial h / \partial x < 0$, and beyond $x_{t,p}$, we have that $\partial h / \partial x > 0$. Hence for every pair (ω, p) admissible under the present consideration, the function $h(x, t)$ has a minimum in the interval $[-1, 1]$ at $x_{t,p}$. Substituting $x_{t,p}$ in (3.7) yields the following bound on the set of all μ 's in C for (3.7). The bound that results is as follows:

$$|\mu| < \frac{p^{1/2} t^{1/2}}{2^{1/p} (p-2)^{1/2-1/p} (1+t)} = \frac{p^{1/2} (\omega-1)^{1/2}}{2^{1/p} (p-2)^{1/2-1/p} \omega}. \quad (3.15)$$

□

For $p = 10$, a graphical illustration of the region $\mathcal{R}(p)$ specified in (3.9)

is given in the shaded area in FIGURE 2 below:

FIGURE 2: The domain $\mathcal{R}(10)$

We close the paper with some remarks.

Remark 3.1: It is readily verified that the right boundary of the region specified in (3.9) and illustrated in FIGURE 2 is, for a fixed p , a strictly decreasing function for $\omega \in (0, 1]$ and a strictly increasing function for $\omega \in [1, 2)$. The behavior of the right boundary for a fixed $\omega \in [1, 2)$ as a function of p is much more intricate and its complete characterization is given in the internal report [8].

Remark 3.2: When $p = 2$, (3.7) reduces to determining the set of all

$\mu \in C$ such that

$$\min_{-1 \leq x \leq 1} (1 + t^2 - 2tx) > \frac{(1-t)^2(1+t)^2}{|t|} |\mu|^2. \quad (3.16)$$

When $-1 < t = \omega - 1 < 0$, (3.16) will be satisfied whenever

$$|\mu| < \frac{(1-\omega)^{1/2}}{2-\omega}.$$

When $0 < t = \omega - 1 < 1$, (3.16) will hold provided

$$|\mu| < \frac{(\omega-1)^{1/2}}{\omega}.$$

References

- [1] G. Alefeld and R. S. Varga. Zur Konvergenz des symmetrischen Relaxationsverfahren. *Numer. Math.*, 25:291–295, 1976.
- [2] A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. *Academic Press*, New-York, 1979.
- [3] L. Chong and D.-Y. Cai. Relationship between eigenvalues of the Jacobi and SSOR iterative matrices for p -weak cyclic matrices. *J. Comp. Math. Coll. Univ.*, 1:79–84, 1985.
- [4] G. Csordas and R. S. Varga. Comparisons of regular splittings of matrices. *Numer. Math.*, 44:23–35, 1984.
- [5] A. Hadjidimos and M. Neumann. A note on the SSOR convergence domain due to Neumaier and Varga. *Lin. Alg. Appl.*, 107:207–217, 1988.
- [6] A. Hadjidimos and M. Neumann. Precise domains of convergence for the block SSOR method associated with p -cyclic matrices. *BIT*, 29:311–320, 1989.
- [7] A. Hadjidimos and M. Neumann. Convergence domains of the SSOR method for generalized consistently ordered matrices. *J. Comp. Appl. Math.*, 33:35–52, 1990.
- [8] A. Hadjidimos and M. Neumann. Domains of dominant convergence of the SSOR method over the SOR method. *Technical Report*, Dept. Comp. Sci., Purdue Univ., West Lafayette, Indiana, 1992.
- [9] W. Kahan. Gauss-Siedel Methods for Solving Large Systems of Linear Equations. *Doctoral Thesis*, Univ. of Toronto, Toronto, 1958.
- [10] U. Kulisch. Über reguläre Zerlegungen von Matrizen und einige Anwendungen. *Numer. Math.*, 11:444–449, 1968.
- [11] A. Neumaier and R. S. Varga. Exact convergence and divergence domains for the symmetric successive overrelaxation iterative (SSOR) method applied to H-matrices. *Lin. Alg. Appl.*, 58: 261–272, 1984.
- [12] M. Neumann. The Kahan S.O.R. convergence bound for nonsingular and irreducible M-matrices. *Lin. Alg. Appl.*, 39:205–222, 1981.

- [13] M. Neumann. On bounds for the convergence of the SSOR method for H-matrices. *Lin. Multilin. Alg.*, 15:13–21, 1984.
- [14] M. Neumann. Neighborhoods of dominant convergence for the SSOR method. *SIAM J. Alg. Discrete Methods*, 7:551–559, 1986.
- [15] N. K. Nichols and L. Fox. Generalized consistent ordering and the optimum successive over-relaxation factor. *Numer. Math.*, 13:425–433, 1969.
- [16] D. O. Tall. Functions of a Complex Variable, Vol. II. *Library of Mathematics*, Routledge Kegan Paul Ltd., London, 1970.
- [17] R. S. Varga. Matrix Iterative Analysis. *Prentice Hall*, Englewood Cliffs, N.J., 1962.
- [18] R. S. Varga, W. Niethammer and D.-Y. Cai. p -cyclic matrices and the symmetric successive overrelaxation method. *Lin. Alg. Appl.*, 58:425–439, 1984.
- [19] H. Wielandt. Unzerlegbare, nicht negative Matrizen. *Math. Z.*, 52:642–648, 1950.
- [20] Z. Woźnicki. Two-sweep iterative method for solving large linear systems and their applications to the numerical solution of multi-group multi-dimensional neutron diffusion equation. *Institute of Nuclear Research*, Technical Report No. 1447/CYFRONET/PM/A, Warszawa, Poland, 1973.
- [21] D. M. Young. Iterative Solutions of Large Linear Systems. *Academic Press*, N.Y., 1971.